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BENDING OF A BUILT-UP SANDWICH BEAM

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Abstract—A built-up sandwich beam is modeled by two thin elastica of facing material affixed to a series of equally spaced rigid separation blocks. The large-deformation equations under pure bending are formulated and solved by perturbations and numerical integrations. The results show the non-linear phenomena of critical load, snap buckling, and hysteresis. A global critical moment is defined.

INTRODUCTION

Due to their light weight and relative strength, composite beams have become extremely important elements in modern structures and machines. One type of widely used composite is the sandwich plate which is composed of two strong, thin elements of facing material attached to a low density core or separated by spacers. Although the sandwich construction has been in use for some time (Marshall, 1982) no previous theoretical work can be found on its behavior, in particular at large deformations.

This paper considers an ideal built-up beam shown in Fig. 1(a). Two thin, strong elastic sheets are glued to evenly spaced rigid blocks. The sheets are thin enough such that they can be considered inextensible elastica (Love, 1944). The behavior of the system under pure bending moments is to be studied.

FORMULATION

Since the built-up beam is under pure bending, the deformation can be described by one representative bent segment shown in Fig. 1(b). Let $2L$ be the lengths of the free elastica elements and λL be the separation distance of the undeformed elastica. By symmetry only the right half needs to be studied.

Due to the pure applied moment, M', the forces on the elastica elements at the midpoints and thus along each element, must be horizontal, equal and opposite. The bottom element shown is under tensile horizontal force F' and the top element is under compression. Let s' be the arc length from the midpoint of the lower element and θ be the local angle of

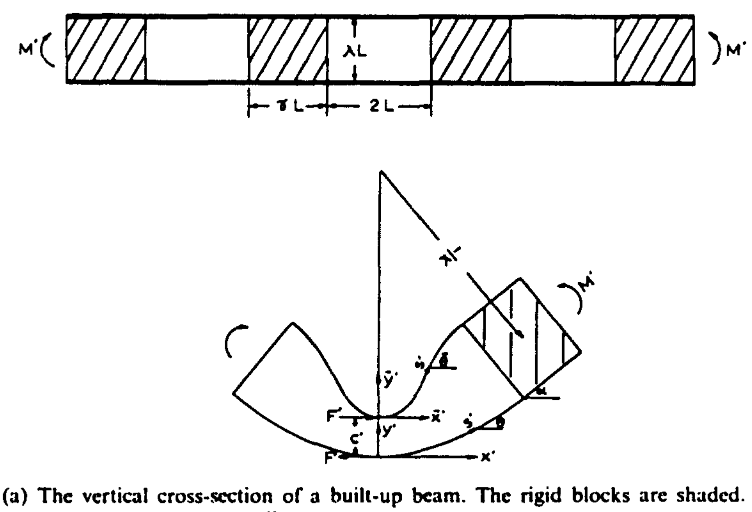

Fig. 1. (a) The vertical cross-section of a built-up beam. The rigid blocks are shaded. (b) The coordinate system on one segment.

inclination. Considering a local moment balance on a small segment ds', the elastica equation (Frisch-Fay, 1962) is

$$EI\frac{d^2\theta}{ds'^2} = F'\sin\theta. \tag{1}$$

Here EI is the flexural rigidity of the elastic sheet. The lengths are normalized by L, the forces by EI/L^2 and the primes are dropped. Equation (1) then becomes

$$\frac{d^2\theta}{ds^2} = F\sin\theta. \tag{2}$$

The Cartesian coordinates (x, y) for the lower sheet can be obtained from

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta. \tag{3}$$

The boundary conditions are

$$x(0) = y(0) = 0, \quad \theta(0) = 0 \tag{4}$$

$$\theta(1) = \alpha. \tag{5}$$

Here α is the half angle of one periodic segment bent by the moment M'. Let an overbar denote the upper elastica element. The governing equations are similar

$$\frac{d^2\bar{\theta}}{ds^2} = -F\sin\bar{\theta} \tag{6}$$

$$\frac{d\bar{x}}{ds} = \cos\bar{\theta}, \quad \frac{d\bar{y}}{ds} = \sin\bar{\theta} \tag{7}$$

$$\bar{x}(0) = \bar{y}(0) = 0, \quad \bar{\theta}(0) = 0 \tag{8}$$

$$\bar{\theta}(1) = \alpha. \tag{9}$$

The unknown force F is obtained by the constraint on the lateral displacement due to the rigid block

$$\bar{x}(1) + \lambda\sin\alpha = x(1). \tag{10}$$

The moment, normalized by EI/L^3 is then

$$M = F\lambda\cos\alpha + \frac{d\theta}{ds}(1) + \frac{d\bar{\theta}}{ds}(1) = CF + \frac{d\theta}{ds}(0) + \frac{d\bar{\theta}}{ds}(0). \tag{11}$$

Here CL is the gap width at the symmetry line. Equations (2)–(11) are to be solved for a given geometry and a given α or M. Due to the high degree of nonlinearity, the analytical solution in closed form is not possible.

PERTURBATION SOLUTION FOR SMALL α

Let $\alpha \equiv \varepsilon^2 \ll 1$. An order-of-magnitude analysis suggests the following expansions for both lower and upper elastica elements:

$$\theta = \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3\theta_3 + \cdots \tag{12}$$

$$F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots \tag{13}$$

$$x = s + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots \tag{14}$$

$$y = \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \tag{15}$$

Note the absence of the O(ε) term in the expansion of x. Such a term can be shown to be zero if included. Equations (2)–(10) give the leading orders

$$\frac{d^2\theta_1}{ds^2} = F_0\theta_1, \quad \frac{d^2\bar{\theta}_1}{ds^2} = -F_0\bar{\theta}_1 \tag{16}$$

$$\theta_1(0) = \theta_1(1) = \bar{\theta}_1(0) = \bar{\theta}_1(1) = 0 \tag{17}$$

$$\frac{dx_2}{ds} = -\frac{\theta_1^2}{2}, \quad \frac{d\bar{x}_2}{ds} = -\frac{\bar{\theta}_1^2}{2}, \quad x_2(0) = \bar{x}_2(0) = 0 \tag{18}$$

$$\bar{x}_2(1) + \lambda = x_2(1) \tag{19}$$

$$\frac{dy_1}{ds} = \theta_1, \quad \frac{d\bar{y}_1}{ds} = \bar{\theta}_1, \quad y_1(0) = \bar{y}_1(0) = 0. \tag{20}$$

Equations (16) and (17) yield

$$\theta_1 = 0. \tag{21}$$

But for a non-trivial solution as required by equation (19), $\bar{\theta}_1$ is not zero

$$\bar{\theta}_1 = C_1 \sin \pi s, \quad F_0 = \pi^2 \tag{22}$$

where the constant C_1 is determined from solving eqns (18) and (19)

$$x_2 = 0, \quad \bar{x}_2 = \frac{-C_1^2}{4}\left(s - \frac{\sin 2\pi s}{2\pi}\right) \tag{23}$$

$$C_1 = \pm 2\sqrt{\lambda}. \tag{24}$$

Equation (20) gives

$$y_1 = 0, \quad \bar{y}_1 = \frac{C_1}{\pi}(1 - \cos \pi s). \tag{25}$$

The next order equations are

$$\frac{d^2\theta_2}{ds^2} = F_0\theta_2 + F_1\theta_1, \quad \frac{d^2\bar{\theta}_2}{ds^2} = -F_0\bar{\theta}_2 - F_1\bar{\theta}_1 \tag{26}$$

$$\theta_2(0) = \bar{\theta}_2(0) = 0, \quad \theta_2(1) = \bar{\theta}_2(1) = 1 \tag{27}$$

$$\frac{dx_3}{ds} = -\theta_1\theta_2, \quad \frac{d\bar{x}_3}{ds} = -\bar{\theta}_1\bar{\theta}_2, \quad x_3(0) = \bar{x}_3(0) = 0 \tag{28}$$

$$\bar{x}_3(1) = x_3(1) \tag{29}$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}s} = \theta_2, \quad \frac{\mathrm{d}\bar{y}_2}{\mathrm{d}s} = \bar{\theta}_2, \quad y_2(0) = \bar{y}_2(0) = 0. \tag{30}$$

The solutions to eqns (26) and (27) are

$$\theta_2 = \frac{\sinh \pi s}{\sinh \pi}, \quad \bar{\theta}_2 = C_2 \sin \pi s - s \cos \pi s \tag{31}$$

$$F_1 = \frac{-2\pi}{C_1}. \tag{32}$$

Using eqns (28)–(30) it can be found that

$$x_3 = 0, \quad \bar{x}_3 = \frac{C_1}{4\pi} s(1 - \cos 2\pi s), \quad C_2 = -\frac{1}{2\pi} \tag{33}$$

$$y_2 = \frac{(\cosh \pi s - 1)}{\pi \sinh \pi}, \quad \bar{y}_2 = \frac{1}{2\pi^2}(1 - \cos \pi s) - \frac{s \sin \pi s}{\pi}. \tag{34}$$

For F_2 a solution for $\bar{\theta}_3$ is needed from

$$\frac{\mathrm{d}^2\bar{\theta}_3}{\mathrm{d}s^2} + \pi^2\bar{\theta}_3 = \frac{\pi^2}{6}\bar{\theta}_1^3 - F_1\bar{\theta}_2 - F_2\bar{\theta}_1 = \frac{-\pi^2 C_1^3}{24} \sin 3\pi s + F_1 s \cos \pi s + \left(\frac{\pi^2 C_1^3}{8} - F_1 C_2 - F_2 C_1\right) \sin \pi s. \tag{35}$$

The solution is

$$\bar{\theta}_3 = \frac{C_1^3}{192} \sin 3\pi s + \frac{F_1}{4\pi} s^2 \sin \pi s - \frac{1}{2\pi}\left(\frac{\pi^2 C_1^3}{8} - F_1 C_2 - F_2 C_1 - \frac{1}{2\pi}\right) s \cos \pi s + C_3 \cos \pi s + C_4 \sin \pi s. \tag{36}$$

The boundary conditions $\bar{\theta}_3(0) = \bar{\theta}_3(1) = 0$ dictate

$$F_2 = \frac{\pi^2 C_1^2}{8} - \frac{1}{2\pi C_1} - \frac{1}{C_1^2}. \tag{37}$$

The normalized maximum local moment experienced is

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}s}(0) = \pi C_1 \varepsilon - \tfrac{3}{2}\varepsilon^2 + \mathrm{O}(\varepsilon^3). \tag{38}$$

The horizontal force is

$$F = \pi^2 - \frac{2\pi}{C_1}\varepsilon + \left(\frac{\pi^2 C_1^2}{8} - \frac{1}{2\pi C_1} - \frac{1}{C_1^2}\right)\varepsilon^2 + \mathrm{O}(\varepsilon^3). \tag{39}$$

Equation (11) gives the moment

$$M = (F_0+\varepsilon F_1+\varepsilon^2F_2)\lambda \cos\alpha+\varepsilon\frac{d\theta_1}{ds}(1)+\varepsilon^2\frac{d\theta_2}{ds}(1)$$

$$+\varepsilon\frac{d\bar{\theta}_1}{ds}(1)+\varepsilon^2\frac{d\bar{\theta}_2}{ds}(1)+O(\varepsilon^3)$$

$$= \frac{\pi^2C_1^2}{4}-\frac{3\pi}{2}C_1\varepsilon+\left(\frac{\pi^2C_1^4}{32}-\frac{C_1}{8\pi}+\pi\coth\pi+\frac{5}{4}\right)\varepsilon^2+O(\varepsilon^3). \tag{40}$$

The constant C_1 is nonuniquely related to λ by eqn (24). The work done is

$$W = \int_0^\alpha M\,d\alpha = \frac{\pi^2C_1^2}{4}\alpha-\pi C_1\alpha\sqrt{\alpha}+\left(\frac{\pi^2C_1^4}{32}-\frac{C_1}{8\pi}+\pi\coth\pi+\frac{5}{4}\right)\frac{\alpha^2}{2}+O(\alpha^{5/2}). \tag{41}$$

Note the work done for the same deflection α is larger for negative C_1 when the elastic sheets bulge out from both sides than that for positive C_1, when they bend towards the same side. Therefore, the positive sign should be taken in eqn (24) since it leads to an equilibrium with lower strain energy, and thus is more likely to happen.

The minimum gap width is then

$$C = \frac{1}{F}\left[M-\frac{d\theta}{ds}(0)-\frac{d\bar{\theta}}{ds}(0)\right]. \tag{42}$$

NUMERICAL INTEGRATION

For α not small eqns (2)-(10) need to be integrated numerically. The process can be simplified as follows. Let

$$s = \frac{r}{\sqrt{F}},\quad x = \frac{\xi}{\sqrt{F}}. \tag{43}$$

The governing equations become

$$\frac{d^2\theta}{dr^2} = \sin\theta,\quad \frac{d\xi}{dr} = \cos\theta \tag{44}$$

$$\frac{d^2\bar{\theta}}{dr^2} = -\sin\bar{\theta},\quad \frac{d\bar{\xi}}{dr} = \cos\theta \tag{45}$$

$$\theta(0) = \xi(0) = \bar{\theta}(0) = \bar{\xi}(0) = 0. \tag{46}$$

For given $d\theta/dr(0)$ and a guessed $d\bar{\theta}/dr(0)$ eqns (44)–(46) are integrated by the Runge–Kutta–Fehlberg algorithm until $\theta = \bar{\theta}$, say at $r = r^*$. At this point the value of

$$\lambda^* \equiv \frac{\xi(r^*)-\bar{\xi}(r^*)}{r^*\sin(\theta(r^*))} = \frac{x-\bar{x}}{\sin\alpha} \tag{47}$$

is noted. The initial guess $d\bar{\theta}/dr(0)$ is adjusted such that λ^* is equal to the prescribed λ. Then

$$F = (r^*)^2 \tag{48}$$

$$\frac{d\theta}{ds}(0) = r^*\frac{d\theta}{dr}(0),\quad \frac{d\bar{\theta}}{ds}(0) = r^*\frac{d\bar{\theta}}{dr}(0) \tag{49}$$

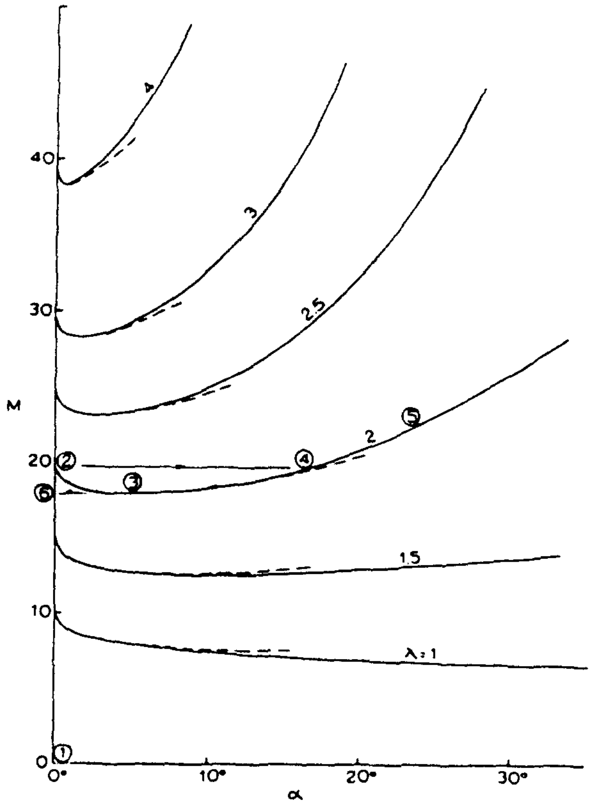

Fig. 2. The normalized moment M vs turning angle α for various λ. Dashed lines are from eqn (40).

$$M = \lambda(r^*)^2 \cos(\theta(r^*)) + r^*\left[\frac{d\theta}{dr}(r^*) + \frac{d\bar{\theta}}{dr}(r^*)\right]. \tag{50}$$

The deformed shape can be determined by using eqns (48) and (49) and integrating eqns (2)–(4) and (6)–(9) for the coordinates (x, y).

DISCUSSION OF RESULTS

Figure 2 shows the important relation between the applied moment M and the turning angle α for various constant geometric ratios λ. The perturbation solution agrees well with exact numerical results for small α. As expected the moment is higher for higher λ (more closely spaced supports). It can be noted however, unlike the pure bending of a single elastic beam, there exists a critical moment below which there is no deformation. This critical moment, from eqn (40), is $\pi^2\lambda$. Furthermore, it is noted that the moment–angle curves at low α have negative slope. This means as soon as the system buckles, it is unstable if a moment is prescribed.

Take for example the case when $\lambda = 2$. (The length of the free elastic segment equal to the separation by the rigid supports.) When moment M is increased from zero at State ①, deformation does not begin until the critical moment ($2\pi^2$) is reached at State ②. Then the system suddenly collapses to State ④ at the same M. If the moment is further increased the system stiffens and the system deforms gradually to State ⑤. The unloading path is different. As the moment decreases the deformation follows States ⑤ and ④ until State ③. The system snaps to the undeformed configuration at State ⑥ with a further decrease in M. Figure 3 shows the configurations of the various states which correspond to those shown in Fig. 2. This snapping hysteresis phenomenon is most prominent at low λ. In fact for $\lambda = 1$ the system offers no resistance at all after buckling. Structurally, high λ geometries (closely spaced supports) offer much better resistance and postbuckling stability.

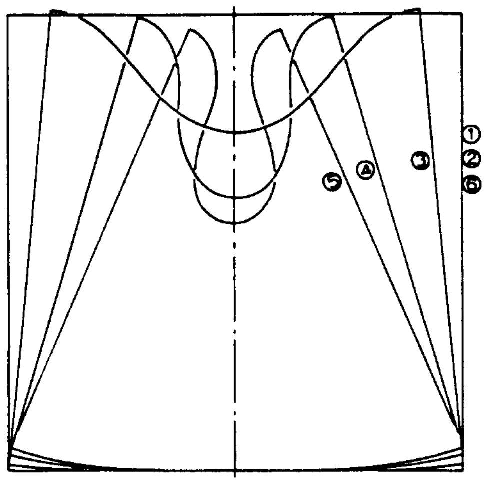

Fig. 3. Deformed configurations for $\lambda = 2$: ① $M = 0$; ② $M = 19.74$; ③ $M = 19.74$; ④ $M = 22.21$; ⑤ $M = 17.82$; ⑥ $M = 17.82$.

The cusps at the critical moments (e.g. at State ②) indicate the system is imperfection sensitive there. Geometric or material imperfection would easily round the cusp and lower the effective critical moment. A more appropriate index for buckling is perhaps the minimum moment (e.g. at State ③) which is called the global critical moment. Below this global critical moment M_{gc}, the only equilibrium state is the undeformed state. it is also relatively insensitive to imperfections. Figure 4 shows M_{gc} increases almost linearly with λ. Below $\lambda = 1.2$, M_{gc} is undefined since the system collapses completely and the two elastic sheets touch at $s = 0$ before reaching a minimum.

Figure 5 shows the normalized lateral force F experienced by the elastic sheets. Similar to the applied moment, the force first decreases then increases with α. The force and applied moment are related by eqn (11), in which the gap with C is decreasing and local moment

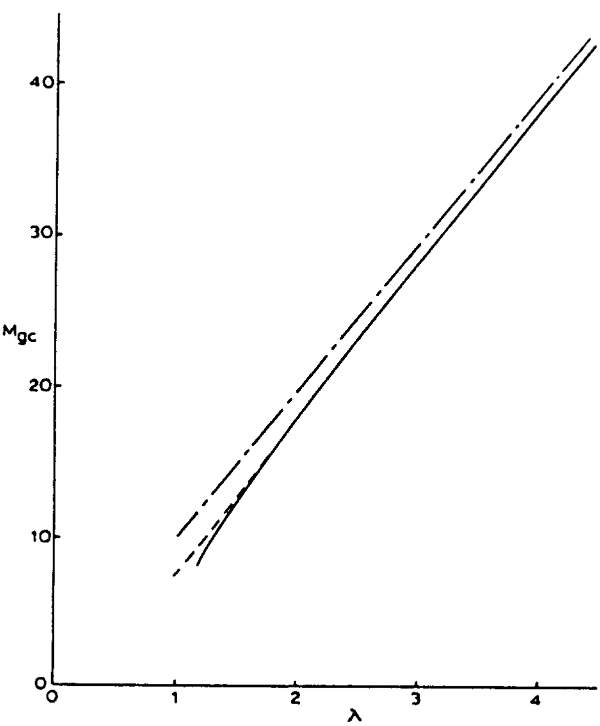

Fig. 4. Global critical moment M_{gc} as a function of λ: —·—·—, $M = \pi^2\lambda$; — — —, from eqn (40); ———, numerical.

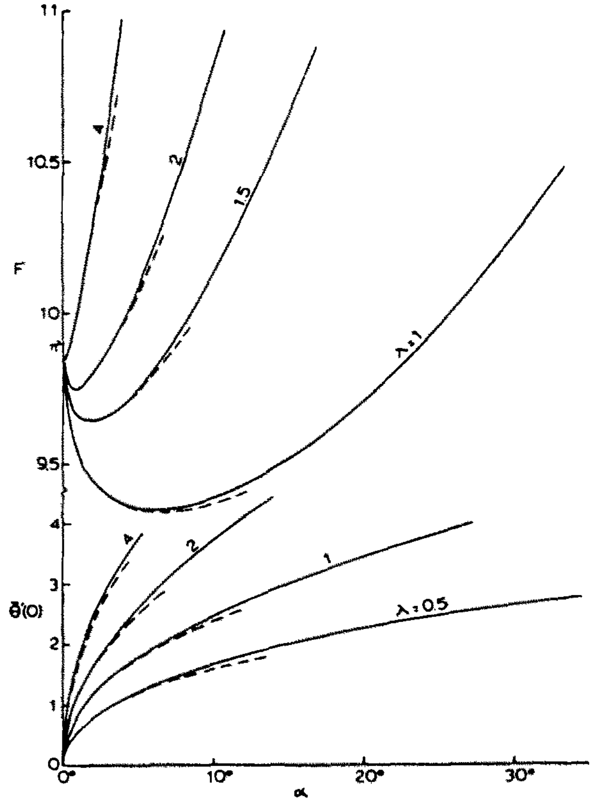

Fig. 5. Horizontal force F and maximum local moment $\bar{\theta}'(0)$ as a function of α for various λ. Dashed lines are from eqn (38) or eqn (39).

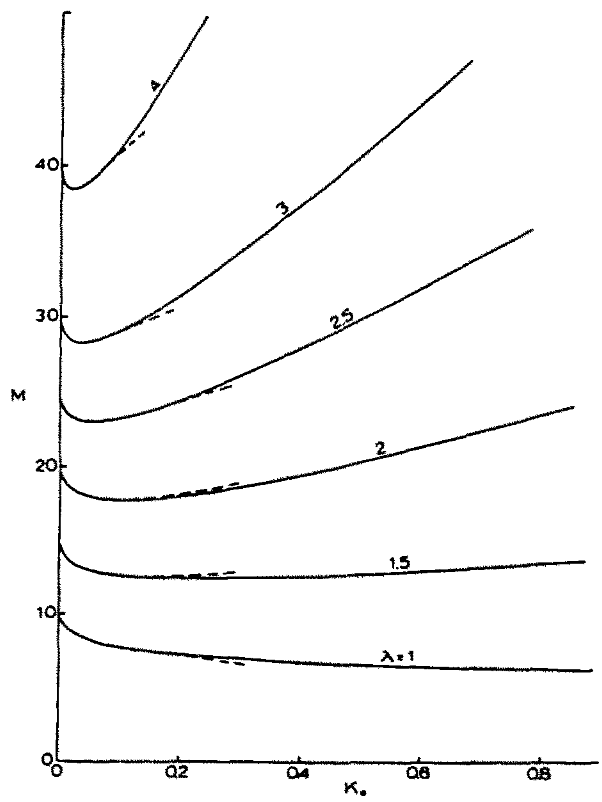

Fig. 6. Moment–curvature relation. Dashed lines are from eqns (40) and (52).

increasing with α. Also shown in Fig. 5 is the maximum normalized local moment $\bar{\theta}'(0)$ which increases with α. Both F and $\bar{\theta}'(0)$ are useful in the design of the sandwich system.

The axial length of the rigid block γL enters through the normalized curvature

$$\kappa = \left(\frac{\gamma}{2 \tan \alpha} + \frac{\bar{x}(1)}{\sin \alpha} + \frac{\lambda}{2}\right)^{-1}. \tag{51}$$

For small α or small κ

$$\alpha = \varepsilon^2 = \left(1 + \frac{\gamma}{2}\right)\kappa\left(1 - \frac{\lambda}{2}\kappa + \frac{\sqrt{\lambda}}{\pi}\sqrt{\left(1 + \frac{\gamma}{2}\right)}\kappa^{3/2} + \cdots\right). \tag{52}$$

Let κ_0 indicate the curvature for $\gamma = 0$. The moment–curvature relation, important in non-linear beam theory, is shown in Fig. 6. The character of the curves is similar to that of the moment–end angle relation, showing critical moment and instability. An empirical approximation for higher λ values for the $M - \kappa_0$ curve would be a vertical straight line from zero to a value less than M_{gc}, then another straight line of positive slope. As a system, this behavior is similar to that of a rigid–plastic beam with strain hardening. When γ is not zero, the abscissa is given by

$$\kappa = \left(\frac{\gamma}{2 \tan \alpha} + \frac{1}{\kappa_0}\right)^{-1}. \tag{53}$$

The interesting behavior of a built-up sandwich beam under pure bending has now been predicted theoretically. It is hoped that this paper will lead to other research, especially well-designed experiments to supplement the important theory of sandwich construction.

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